## Second order differential eqn.

$M(tx, ty) = t^\alpha M(x,y)$

Homogeneous function

Homogenous eqn → A Linear $n^{th}$ order DE of the form

$$\rightarrow \quad a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}(y)}{dx^{n-1}} + ----$$

$$a_1(x)\frac{dy}{dx} + a_0 y = 0 \quad —— ⑥$$

is homogeneous

where as

$$\rightarrow \quad a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + --- a_1(x)\frac{dy}{dx}$$

$$+ a_0 y = g(x)$$

$$g(x) \neq 0 \quad —— ⑦$$

is non-homogeneous

$a_i(x), g(x)$ continuous and $a_n \neq 0$ for every $x$.

ex $3y''' + 2y'' + 6x = 0$ Non-homogeneous
$3y''' + 2y'' + 6y = 0$ homogeneous.

SUPERposition principle Homogeneous eqn.

let $y_1, y_2, ---- y_k$ be soln of homogenous eqn (6) on I, then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + ---- + c_k y_k(x)$$

where $c_1, c_2 ---- c_k$ are constants, is also a soln on the interval.

Proof $K = 2$

Let $\overset{\textcircled{6}}{y_1}, \overset{\textcircled{6}}{y_2}$ are soln.

$\overset{\textcircled{6}}{c_1 y_1 + c_2 y_2}$ is also soln.

Hence proved.

ex The function $y_1 = x^2$, $y_2 = x^2 \ln x$ are both soln, of ODE

$$x^3 y''' - 2xy' + 4y = 0$$ on

$(0, \infty)$. Then by superposition principle

i.e. $y = c_1 x^2 + c_2 x^2 \ln(x)$ is also a soln.

ex $f_1(x) = \sin x$, $f_2(x) = \sin x \cos x$

LD or LI.

LD ↓ Linearly Dependent

LI → Linearly Independent

Wrounsicion Suppose each of the function $f_1(x), f_2(x) ---, f_n(x)$ posses atleast $n-1$ derivatives. The determinant

$$W(f_1, f_2 --- f_n) = \begin{vmatrix} f_1 & f_2 & ---- & f_n \\ f_1' & f_2' & ---- & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{n-1} & f_2^{n-1} & & f_n^{n-1} \end{vmatrix}$$

is called wronskion

if $y_1, y_2 --- y_n$ are soln, of homogeneous linear Dependent on I, they they are LI iff $W(y_1, y_2 -- y_n) \neq 0$.

Lemma 1 if $y_1(x)$ and $y_2(x)$ are two soln of $y'' + p(x)y' + Q(x)y = 0$ on $[a, b]$, there their $W = w(y_1, y_2)$ is either identically zero or never zero.

eg $f(t) = t^3$

$$g(t) = |t|^3 = \begin{cases} t^3 & t > 0 \\ -t^3 & t < 0 \end{cases}$$

If $t > 0$

$$W(f, g) = \begin{vmatrix} t^3 & t^3 \\ 3t^2 & 3t^2 \end{vmatrix}$$

$$= 0$$

If $t < 0$ $W(f, g) = 0$

$\Rightarrow W = 0$

But $f(t)$ and $g(t)$ are LD.

$f(t) = c\,g(t)$. on $(-\infty, \infty)$

either on $(-\infty, 0)$, on $(0, \infty)$

they are LI $\forall\, t \in (-\infty, \infty)$ because we ~~do cant f~~ cann't find a such c'

⇒ These never be soln of homogeneous eqn.

ex let $f(x) = x|x|$
$g(x) = x^2$.

**Theoram**

If f and g be any two differentiable function on I. ~~If~~ $W(x) = 0$ for atleast one ~~x =~~ $x \in I$ then f, g are LI on I.

$g, f \ (a,b) \to R$

or if f and g ~~are~~ are LI on I
then $w(~~x~~) \neq$, ~~[illegible]~~ $\angle R$
[$\exists x$ ]

## Fundamental set of soln

Any set $y_1, y_2 - - - y_n$ of n L.I. soln of homo $n^{th}$ order linear one is called

Th ~~existance from of fundamental d.set~~

There always exist a fundamental set for
L-H-$n^{th}$ order.

Def General soln of homo eqn

If $y_1, y_2, \dots, y_n$ forms a fundamentul set of soln then

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

where $c_i$'s are constants is a general soln

**ex** $y'' - 9y = 0$, $y_1(x) = e^{3x}$
$y_2(x) = e^{-3x}$
are soln.

$$y = c_1 y_1 + c_2 y_2$$

**Non- homogeneous eqn**

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots a_0(x) y = g(x)$$

$$g(x) \neq 0$$

**Particular solution**

Any function (any soltn) $y_p$ free of arbitrary parameters.

**Thm** **General solution**
Let $y_p$ be particular solution of ⑦ (NHLDE) on I. Let
$y_1, y_2 \cdots y_n$ be fundamental set of soln, of associated homo eqn then

general soln of ⑦ on I is

$$y = \underbrace{c_1y_1 + c_2y_2 + \cdots c_ny_n}_{} + y_p$$

$c_i$ → arbitiary constants.

$$\underset{\downarrow \text{operator.}}{\mathcal{L}}(y) = g(x)$$

complementary function (soln)

$$y = y_c + y_p$$

**Thm** Superposition principle.

Suppose $y_{p_i}$ denotes particular soln of corresponding DE.

$$a_ny^n + a_{n-1}y^{n-1} + \cdots a_ny = g_i(x)$$

$$i = 1, 2, \cdots K$$

then $y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots y_{p_k}(x)$ is particular solution of

$$a_ny^n + a_{n-1}y^{n-1} + \cdots a_0y = g_1(x) + g_2(x) + \cdots g_k(x)$$

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ex $L(y) = y'' - 3y' + 4y = 2e^{2x} + 2xe^{x} - e^{x}$

$$L = \left(a_m \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots\right)$$

ex $$2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx}$$

$$\left(\frac{2d^2}{dx^2} + 3x\frac{d}{dx}\right)y$$

$$L(y)$$

$y_{p_1}$ of $L(y) = 2e^{2x}$

$y_{p_2}$ of $L(y) = 2xe^{x}$

$y_{p_3}$ of $L(y) = -e^{x}$

Solution techniques

1) ~~Homo~~ L-2nd order
2) Constant coefficient

$\propto$)

2) $ay' + by = 0$

$y' = ky, \quad k = -b/a$

$y = e^{kx}$ or a $e^{mx}$

$y = ke^{kx}$
$= ky$

Ex

$y' = ky, \; y = e^{mx}$

$\downarrow$

$me^{mx} = \frac{-b}{a} e^{mx}$

$= (am+b)\, e^{mx} = 0$

$m = -b/a$

$\boxed{y = e^{-b/a\, x}} \Rightarrow y = e^{mx}$

Ref

$y_2 = uy_1, \; P = \frac{-b}{a}, \; Q = c/a$

$ay_2'' + by_2' + cy_2 = 0$

$y_2' = uy_1' + u'y_1$

$u(\cancel{y_1'' + Py_1' + Qy_1} \to 0) +$

$u'(2y_1' + Py_1) + u''y_1 = 0$

$ay'' + by' + cy = 0$

put $y = e^{mx}$

$\frac{am^2 + bm + c = 0}{e^{-mx}}$

$\boxed{am^2 + bm + c = 0} \begin{cases} m_1 \\ m_2 \end{cases}$

$\downarrow$

auxilliary eqn.

$m_1, m_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

**Case-I** Real + distinct

$\underbrace{y_1 = e^{m_1 x}, \; y_2 = e^{m_2 x}}$

LI on $(-\infty, \infty)$

$\downarrow$

Fundamental set

General soln

$\Rightarrow \boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

**Case-II** Real + equal

$m_1 = m_2, \; b^2 = 4ac$

$y_1 = e^{-\frac{b}{2a} x}$

$y_2 = u(x)\, y_1(x)$

$y_2 = x\, y_1(x)$

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$u' = w, u'' = w'$

$(2y_1' + Py_1)w + yw' = 0$

$$\frac{w'}{w} = \left(\frac{2y_1'}{y_1} + P\right)$$

$$w = \frac{c_1}{y_1^2} e^{-\int P dx}$$

$$u' = \frac{c_1}{y_1^2} e^{2m_1 x}$$

$u = c_1 x + c_2$

$u = x$ if we choose $c_1 = 1, c_2 = 0$

$$y_1 = e^{-\frac{b}{2a}x}$$

$$y_2 = x e^{-\frac{b}{2a}x}$$

$y = c_1 y_1 + c_2 x y_1(x)$

$$\boxed{y = (c_1 + c_2 x) e^{-\frac{b}{2a}x}}$$ general eqn.

**Case-3** $m_1 = \alpha + i\beta$

$m_2 = \alpha \pm i\beta$

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$$

$$y = c_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + c_2 e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

~~$y = e^{\alpha x}$~~ $= \boxed{y = e^{\alpha x}(c_1 \cos\beta x + i c_2 \sin\beta x)}$

General soln.

**ex-1** $y'' + 4y' + 3y = 0$, $y(0) = 2$
$y'(0) = -1$

$\Rightarrow$ if soln is in form of $e^{mx}$:

$(m^2 + 4m + 3)e^{mx} = 0$

$m^2 + 4m + 3 = 0$

$m = -3, -1$

$$\boxed{y = c_1 e^{-3x} + c_2 e^{-x}.}$$

$y(0) = 2$

$2 = c_1 + c_2$

$y'(0) = -1 \Rightarrow -1 = -3c_1 - c_2$

$-1 = -3c_1 - 2 + c_1$

$c_1 = -\frac{1}{2}$, $c_2 = \frac{5}{2}$

$\therefore$ general solution.

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$$\boxed{y = -\frac{1}{2}e^{-3x} + \frac{5}{2}e^{-x}}$$

&

particular soln.

ex

$y'' - 10y' + 25y = 0$

$m^2 - 10m + 25 = 0$

$m = 5, 5$

$y = (c_1 + c_2 x)e^{5x}$

ex

$y''' + 3y'' - 4y = 0$

$m^3 + 3m^2 - 4 = 0$

~~$m = 0, \quad m^2 - 3m - 4 = 0$~~

~~$(m-4)(m+1) = 0$~~

~~$m = 4, -1.$~~

$m = 1$

$$\begin{array}{r|l} & m^2 + 4m + 4 \\ \hline m-1 & m^3 + 3m^2 - 4 \\ & \underline{m^3 - m^2} \\ & 4m^2 - 4 \\ & \underline{4m^2 - 4m} \\ & 4m - 4 \end{array}$$

$(m+2)^2 = 0$

$m = -2, -2$

$y = c_1 e^{x} + (c_2 + c_3 x) e^{-2x}$

ex-4 $\quad y'''' + 2y'' + y = 0$

$m^4 + 2m^2 + 1 = 0$

$(m^2+1)^2 = 0$

$m = i, -i, i, -i$

~~$y = c_1 e^{ix} + c_2 x$~~

$y = (c_1 + c_2 x)\, e^{ix} + (c_3 + c_4 X)\, e^{-ix}.$

## Reduction of order

General second ODE → $F(x, y, y', y'') = 0$

we consider two special types of ~~sec~~ second ODE.

I) Dependent variable is missing.

y is not explicit present, then eqn can be written as.

$$F(x, y', y'') = 0$$

then, we substitute

$y' = p, \quad y'' = \frac{dp}{dx}$

$F(x, p, p') = 0$

ex

$xy'' + y' = 4x$

$y'' = p_x \,, \; y' = p$

$xp' + p = 4x$

$\Rightarrow \quad \frac{dp}{dx} + \frac{p}{x} = 4$

$e^{\int 1/x \, dx} = x$

$px = 2x^2 + c$

$\frac{dy}{dx} = 2x + \frac{c}{x}$

$\boxed{y = x^2 + c_1 \ln x + c_2}$

II) Independent variable is missing.

$F(y, y', y'') = 0$

$y' = p, \quad \frac{dy}{dx} = p$

$\frac{d^2y}{dx^2} = \frac{dp}{dy} \frac{dy}{dx}$

$= p \frac{dp}{dy}$

$$F\left(y, P, \frac{dP}{dy}\right) = 0.$$

ex

$$y'' + K^2 y = 0$$

ex

$$y\,y'' + (y')^2 = 0$$

$$y\,\cancel{P}\frac{dP}{dy} + P^{\cancel{2}} = 0$$

$$\int -\frac{dP}{P} = \int \frac{dy}{y}$$

$$-\ln P = \ln y + c$$

$$\ln Py = c$$

$$\int y\frac{dy}{dx} = \int e^c$$

$$\frac{y^2}{2} = x e^c + c_2$$

$$\boxed{y^2 = c_1 x + c_2}$$

→ Homogenous linear second order ODE.

$y_1$, $y_2$ → $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ — ①

Non-trivial solution of $y_1$ — ①

$y_1, y_2$ are LI on $\underline{I}$.

$\frac{y_2}{y_1} = u(x)$

$F(f(x)y'', g(x)y', g(x)y)$

$y_2 = u(x)\, y_1$

$u(x) = ?$

ex

$y_1 = e^x$

$y'' - y = 0 \;\begin{matrix} y_1 \\ y_2 \end{matrix}$

$y_2 = ?$

$y_2 = u(x)\, y_1$

$y_2 = ue^x$

$y_2' = ue^x + u'e^x$

$y_2'' = ue^x + u_x e^x + u_x x e^x$

$$= ue^{x} + 2uxe^{x} + 4xxe^{x}$$

$$y_2'' = e^{x}(u + 2u' + u'')$$

$$y_2'' - y_2 = 0$$

$$e^{x}(u + 2u' + u'') - e^{x}u = 0$$

$$2u' + u'' = 0$$

$$u' = p$$

$$u'' = p'$$

$$2p + p' = 0$$

$$2p = -\frac{dp}{dx}$$

$$2x + c = -\ln p$$

$$p = c_1 e^{-2x}$$

$$\Rightarrow \frac{du}{dx} = c_1 e^{-2x}$$

$$u = c_2 e^{-2x} + c_3$$

$$y_2 = (ce^{-2x} + c_2)e^{x}$$

~~$= ce^{-x}$~~.

$$= c_1 e^{-x} + c_2 e^{x}$$

$$y_2 = c_1 e^{-x}.$$

Th

Let $f(x)$ and $g(x)$ any two differentiable function on I. and g are linear dependent on I. then

$$w(x)=0 \quad \forall x \in I.$$

Or if $w(x) \neq 0 \ \forall x \in I$ then f & g are LI on I.

$$f = e^{t}$$

ex $f = e^{t}, \quad g = e^{2t}.$

$$w(f,g)\big|_{t_0} = e^{3t_0} \neq 0$$

LI on interval I.

Abel's theoram

If $y_1$ & $y_2$ are soln. of DE

$$y'' + P(t)\,y' + q(t)y = 0$$

Where P, q are contn on open interval I then

$$w(y_1, y_2) = c\, e^{-\int P(t)\,dt}$$

To solve a non-homogenous linear DE

$$a_n y^n + a_{n-1} y^{n-1} + \cdots a_1 y' + a_0 y = g(x) \quad ①$$

1. Find complementary function $y_c$.
2. Find particular soln $y_p$ of ①.
3. $y = y_c + y_p$ is general soln of ①.

By Method of undetermined coefficient superposition.

We develop a method to find $y_p$, motivate by the kinds of function that make up input for $g(x)$.

↳ Ex

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$

$y = y_c + \boxed{y_p}$

$y_c$ = soln of $y'' + 4y' - 2y$ at $g(x) = 0$.

$m^2 + 4m - 2 = 0$

$\frac{-4 \pm \sqrt{24}}{2}$

$m = -2 \pm \sqrt{6}$.

$y_c = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$

Find $y_p$.

$g(x)$ is a quadratic polyno.
let us assume

$$y_p = Ax^2 + Bx + c$$

we seek A, B, C s.t $y_p$ is a soln of given ODE.

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$2A + 4(2Ax+B) - 2(Ax^2+Bx+c) = 2x^2 - 3x + 6$$

$-2A = 2$, $8A - 2B = -3$, $2A + 4B - 2c = 6$

$\boxed{A = -1}$ $\quad -8 - 2B = -3$

$\boxed{B = -3/2}$, $\quad -2 - 6 - 2c = 6$

$-14 = 2c$

$\boxed{c = -7}$

$\rightarrow$ $L(y) = g_1(x) + g_2(x)$

$y_1 \rightarrow L(y) = g_1(x)$

$y_2 \rightarrow L(y) = g_2(x)$

$y = y_1 + y_2$.

| $g(x)$ | forms of $y_p$ |
|---|---|
| $K$ | $A$ |
| $5x+7$ | $Ax + B$ |
| $3x^2+2$ | $Ax^2+Bx+c$ |
| $x^3-x+1$ | $Ax^3+Bx^2+Cx+d$ |
| $\sin ax, \cos ax$ | $A\cos ax + B\sin ax$ |
| $e^{mx}$ | $Ae^{mx}$ or $Axe^{mx}$ |
| ~~ax~~ $(ax+b)\, e^{mx}$ | $(Ax+B)e^{mx}$ |
| $ax^2 e^{mx}$ | $(Ax^2+Bx+C)e^{mx}$ |
| $e^{ax}\sin bx$ | $A\, e^{ax}\cos bx + Be^{ax}\sin(bx)$ |
| $5x^2 \sin bx$ | $(Ax^2+Bx+C)\cos bx +$<br>$+(Ex^2+Fx+G)\sin bx$ |

Not applicable to
$\ln x$, $\frac{1}{x}$, $\tan x$, $\tan^{-1}x$.

Find particular soln of $y'' - 5y' + 4y = 8e^x$.

$y_p(x) = Ae^x$, $m^2 - 5m + 4 = 0$

$y_c = c_1 e^{+4x} + c_2 e^{+x}$

$Ae^x - 5Ae^x + 4Ae^x = 8e^x$

$0 = 8.$

sol $\boxed{y = c_1 e^{4x} + c_2 e^{x}}$

$y_p = Axe^x$

$y_p'' = A[e^x + xe^x + e^x]$

$y_p' = A[xe^x + e^x]$

~~$Axe^x + Ae^x$~~

$Ae^x + A\cancel{xe^x} + Ae^x - 5x\cancel{A}e^x - 5Ae^x$
$+ 4A\cancel{xe^x} = 8e^x$

~~$-4A = 8$~~

$\boxed{\cancel{A = -\frac{8}{3}}}$

$-3A = 8$

$\boxed{A = -\frac{8}{3}}$

The $y_p$ is a set of all ~~LD~~ L.I function that are generate by ~~separate~~ differentiations of $g(x)$. repeated.

→ Method of Variation of parameters.

We examine a method for determined $y_p(x)$ of non-homogenous DE that has in theory, no restrictions on it.

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Solve $a_2 y'' + a_1 y' + a_0 y = g(x)$

I) put $y'' + Py' + Q y = f(x)$

II) $y_c$ of eqn in step 1.

$$y_c = c_1 y_1 + c_2 y_2$$

III) Calculate $W(y_1(x), y_2(x))$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

IV) Calculate $u_1$ & $u_2$ by solving

$$u_1' = \frac{w_1}{w}, \quad u_2' = \frac{w_2}{w}$$

V) $y_p = u_1 y_1 + u_2 y_2$

VI) $y = y_c + y_p$

Consider a linear Second ODE.

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

Standard form

$$y'' + P(x)y' + Q(x)y = f(x) \quad \text{—⑥}$$

we know that $y_c = c_1 y_1(x) + c_2 y_2(x)$

is ~~associated~~ complemented soln of associated homogenous eqn of (6).

$c_1, c_2$ replace by $u_1, u_2$.

$$\boxed{y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)} \quad \text{⑦}$$

$$\left[ \because y_1(x) \quad y_2(x) = u(x)\, y_1(x) \right]$$

substitute ⑦ into (6).

$$u_1[y_1'' + Py_1' + Qy_1] + u_2[y_2'' + Py_2' + Qy_2]$$
$$+ (y_1 u_1'' + u_1' y_1') + (y_2 u_2'' + u_2 y_2') +$$
$$P[y_1 u_1' + y_2 u_2'] + y_1 u_1' + y_2' u_2' = f(x)$$

$$\frac{d}{dx}[y_1 u_1' + y_2 u_2'] + P[y_1 u_1' + y_2 u_2')$$

$$+ y_1' u_1' + y_2' u_2' = f(x).$$

Simce, we want to calculate of $u_1$ & $u_2$

we suppose, $\boxed{y_1 u_1' + y_2 u_2' = 0}$ ✓

⑧ becomes

$$\boxed{y_1' u_1' + y_2' u_2' = f(x)}$$

By Cramers rule, soln of system above can be expressed in terms of determinant

$$u_1' = \frac{w_1}{w}, \quad u_2' = \frac{w_2}{w}$$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$w \neq 0$ $y_1, y_2$ are soln of Homogenous eqn.

$$u_1' = \frac{-y_2 f(x)}{W} \quad , \quad u_2' = \frac{y_1 f(x)}{W}$$

integrate to find $u_1, u_2$

$$y_p = u_1 y_1(x) + u_2(x) y_2(x)$$

ex. $y'' - 4y' + 4y = (x+1)e^{2x}$

$m^2 - 4m + 4\cancel{m}$

$(m-2)^2$

$$y_c = (c_1 + c_2 x)\, e^{+2x}$$

$$W = \begin{vmatrix} \overset{y_1}{e^{+2x}} & \overset{y_2}{xe^{+2x}} \\ +2e^{+2x} & e^{+2x} + 2xe^{+2x} \end{vmatrix}$$

$$= e^{4x}(1+2x) - 2xe^{4x}$$

$$W = e^{4x}$$

$$W_1 = -y_2 f(x)$$
$$= -xe^{2x}(x+1)e^{2x}$$
$$= -x(x+1)e^{4x}$$

$$W_2 = y_1 f(x)$$
$$= e^{2x}(x+1)e^{2x}$$
$$= (x+1)e^{4x}$$

$u_1' = -x(x+1)$ , $u_2' = x+1$

$u_1 = -\frac{x^3}{3} - \frac{x^2}{2}$ $\quad u_2 = \frac{x^2}{2} + x \cancel{+c}$

$$y = (c_1 + c_2 x)e^{2x} + \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}$$

## * → Annihilator approach by Kari

$n^{th}$ order diff. eqn can be written as

$$a_n D^n y + a_{n-1} D^{n-1} \cdot y + \cdots a_1 Dy + a_0 y = g(x) \quad — ①$$

$$\underset{\substack{\downarrow \\ \text{differential operator}}}{D^K} = \frac{d^K}{dx^K}$$

$$L \equiv a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0$$

$$L(y) = f(x)$$

**Annihilator operator** If L is a differential operator will constant coefficients & f is sufficiently differentiable function s.t.

$$L(f(x)) = 0$$

then L is said to ~~to~~ be annihaltor of the f(x).

ex

$$DK = 0$$
$$D^2 x = 0$$
$$D^3 x^2 = 0$$

* $(D-\alpha)^n$ annihates $e^{\alpha x}, xe^{\alpha x}$ ~~,~~ --- $x^{n+1}e^{\alpha x}$.

$$(D-2)e^{2x} = De^{2x} - 2e^{2x}$$
$$= 2e^{2x} - 2e^{2x}$$
$$= 0.$$

$1-5x^2+8x^3$ — $D^4$
$e^{-3x}$ $D+3$

Hari $(D^2 - 2\alpha D + (\alpha^2+\beta^2)]^n$ annhilates

$e^{\alpha x}\cos\beta x, xe^{\alpha x}\overset{\sin\beta x}{\cos\beta x}$ --- $x^{n+1}e^{\alpha x}\cos\beta x$

$(D-a)^2+b^2 \longrightarrow e^{ax}\sin bx$.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + --- a_1 y + a_0 = g(x)$$

$$L(y) = g(x)$$

$$L \equiv a_n D^n + a_{n-1} D^{n-1} + --- a_1 D + a_0$$

$$L(x^2) = 0$$

$$L \equiv D^3$$

Annhilator approach. by Hari.

अजन हैन

ex $y'' + 3y' + 2y = 4x^2$

M-1 $y^{(5)} + 3y^{(4)} + 2y''' = 0$

$$m^5 + 3m^4 + 2m^3 = 0$$

$$m^3(m^2 + 3m + 2) = 0$$

$$m^3(m+2)(m+1) = 0$$

$$m = 0, 0, 0, 2, 1$$

$$c_1 e^{2x} + c_2 e^x + (1 + x + x^3$$

M-2 $y'' + 3y' + 2y = 0$

AE $m^2 + 3m + 2 = 0$

$$m = -2, -1$$

$$y_c = c_1 e^{-2x} + c_2 e^{-x}$$

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$L(4x^2) = 0$

$L \equiv D^3$

$D^3(D^2 + 3D + 2)y = 0.$

$m^3(m^2 + 3m + 2) = 0$

$m = 0, 0, 0, -2, -3$

$$y_c = \underbrace{(c_1 + c_2 x + c_3 x^2)}_{\uparrow y_p} + \underbrace{c_4 e^{-2x} + c_5 e^{-x}}_{\uparrow y_c}$$

By method of undeterminany

$L(y) = g(x)$

Linear initial value problem of order n.

$a_n y^n + a_{n-1} y^{n-1} + \cdots a_1 y' + a_0 y = g(x)$

Existance & Uniqueness ~~of~~ or linear IVP of order n.

Suppose $a_i(x), g(x) \in$ ~~$C(x)$~~ $C(I)$ and $a_n(x) \neq 0$ $\forall x \in I$. Let $x_0 \in I$, then IVP $L(y) = g(x)$, $y^i(x_0) = a_i$, $i = 0, 1 \ldots n-1.$ has a unique soln $y(x)$ $\forall x \in I$.

$[y' = f(x, y)$

**ex** Interval of validity of

$(t^2 - 9)y'' + ty' + 2y = \frac{1}{\cancel{t-3}}$

$y(4) = -2.$

$$y'' + \frac{t}{t^2-9}y' + \frac{2}{t^2-9}y = \frac{1}{t^2-9}.$$

$t = \pm 3$ discontinuous.

$(-\infty,-3),\ \boxed{(3,\infty)}$ | include y.
$(-3,3)$

Shantjindal

ex $(t^2-4t)y'' + 3ty' + 4y = 2,$
$y(3)=0,\ y'(3)=1.$

~~For~~ Find an interval where soln of above eqn is unique.

$(0,4)$

---

Variation of parameters for general $n^{th}$ order differential equation

$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

$L(y) = g(t)$

$L(y) = y^{(n)} + \underline{p_1(t)}y^{n-1} + \cdots p_{n-1}y'(t) + p_n(t)y$
$= g(t)$

**Step-I** Put ① into standard form

**Step-II** find complement function ($y_c$)

$$y_c = c_1 y_1(t) + c_2 y_2(t) + - - - c_n y_n(t).$$

**Step-III** $W(y_1(t), y_2(t) - - - - y_n(t))$

$$= c \exp\left(- \int p_1(t)\, dt\right)$$

Abel's ~~soln.~~ identity

**Step-IV**

Wm: determinant obtained from W by replacing $m^{th}$ column by

$$(0, 0, 0 - - - - 1)$$

V) $$y_p(t) = \sum_{m=1}^{n} y_m(t) \int_0^t \frac{g(s)\, W_m(s)\, ds}{W(s)}$$

Q $y''' - y'' - y' + y = g(t)$

Find $y_p$ in terms of integral.

$$m^3 - m^2 - m + 1 = 0$$
$$m^2(-1+m) + 1(1-m) = 0$$
$$m^2(-1+m) - 1(m-1) = 0$$
$$(m-1)(m^2-1) = 0$$
$$m = 1, \pm 1, -1$$

$$y_1 = e^t, \quad y_2 = te^t, \quad y_3 = e^{-t}$$

$$W(e^t, te^t, e^{-t}) = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & +e^{-t} \end{vmatrix}$$

$\int -(-1)\,dt$

$ce$ $\underline{c = 4}$

$ce^t$

$$W_1 = \begin{vmatrix} 0 & te^t & e^{-t} \\ 0 & (t+1)e^t & -e^{-t} \\ 1 & (t+2)e^t & +e^{-t} \end{vmatrix} = -2t-1$$

$$W_2 = \begin{vmatrix} e^t & 0 & e^{-t} \\ e^t & 0 & -e^{-t} \\ e^t & 1 & +e^{-t} \end{vmatrix} = 2$$

$$W_3 = e^{2t}$$

$$y_p(t) = e^{t}\int_0^t \frac{g(s)(-1-2s)}{4e^s}\,ds + \cdots$$

$$te^{t}\int_0^t \frac{g(s)(2)}{4e^s}ds + e^{-t}\int_0^t \frac{g(s)(e^{2s})}{4e^s}\,ds$$

$$\cancel{\int_0^t \frac{g(s)}{4e^s}\left[-e^{+t} - 2se^{+t} + 2te^{t}\right.}$$

Linear differential eqn with variable coefficient

Cauchy Euler's eqn

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots a_1(x)\frac{dy}{dx}$$

✳ $$+ a_0 y = g(x)$$

where $a_0, a_1, a_2, \ldots a_n$ are constants.

Method of soln

consider a second order eqn

$$ax^2 y'' + bxy' + cy = 0$$

we try a soln of the form $y = x^m$

(I) becomes

$$am(m-1)x^m + bm\,x^m + cx^m = 0$$

$$am(m-1) + bm + c = 0$$

$$am^2 + (b-a)m + c = 0.$$

case-I , distinct real roots.

Let $m_1, m_2$

$$y_1 = x^{m_1}, \quad y_2 = x^{m_2}$$

form fundamental set of soln.

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

case-II

repeated roots.

$$m_1 = -\frac{(b-a)}{2a}$$

$$y_1 = x^{m_1}, \quad y = c_1 x^{m_1} + c_2 x^{m_1} \ln x$$

$$= (c_1 + c_2 \ln x) x^m$$

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$m_1 = \alpha + \beta i, \; m_2 = \alpha - i\beta$

$y = c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta}$

Ex-1 $\quad x^2 y'' - 2xy' - 4y = 0$

$a = 1, \; b = -2, \; c = -4$

$m^2 - 3m - 4 = 0$

$m = -1, 4$

$y'_c = c_1 x^{-1} + c_2 x^4$

ex $\quad x^3 y''' + 5x^2 y'' + 7xy' + 8y = 0$

~~$m^3 + 5m^2 + 2m + 8 = 0$~~

$y = x^m$

$m(m-1)(m-2) + 5(m)(m-1) + 7(m) + 8 = 0$

$(m^2 - m)(m-2) + 5m^2 - 5m + 7m + 8 = 0$

$m^3 - 2m^2 - m^2 + \underline{2m} + 5m^2 + \underline{2m} + 8 = 0$

$\underline{m^3 + 2m^2 + 4m + 8 = 0}$

$m = -2$

$$\begin{array}{r} m^2 + 4 \\ m+2 \,\big)\, m^3 + 2m^2 + 4m + 8 \\ \underline{m^3 + 2m^2} \\ 4m + 8 \end{array}$$

$\underline{m = -2, \; 2i, \; -2i}$

Roots are equal

$$(C_1+C_2x+C_3x^2+---)e^{xm}$$

$$(C_1+C_2\ln x+C_3\ln^2 x+----)x^m$$

Linear
↓ ↓ Homo

→ Method of soln for second order LHDE with variable coffeicient.

$$P(x)\,y''+Q(x)\,y'+R(x)\,y=0 \quad — ①$$

M-I exact form

① is said to be exact if it can be written as ~~(P(x)y)'+~~ $\boxed{(P(x)y')'+(f(x)y)'=0}$ — ②

where f(x) has to be determind in terms of P(x), Q(x), R(x).

Then 2 can be integrated immediately to find the soln.

Aim Find f(x).

$$② \to P(x)y''+P'(x)y'+f(x)y'+f'(x)y=0 \quad —— ②'$$

Comparing ① & ②'

$$P'(x)+f(x)=Q(x) \quad — ③$$
$$f'(x)=R(x) \quad —— ④$$

③ $\Longrightarrow P''(x) + f'(x) = Q'(x)$

$$\boxed{P''(x) + R(x) - Q'(x) = 0} \quad — ⑤$$

necessary and sufficient condtn for exactness of 1.

**ex** $y'' + xy' + y = 0$ — Ⓐ

$P(x) = 1, \; Q(x) = x, \; R(x) = 1$

$0 + 1 - 1 = 0$

write $(y')' + (f(x) \cdot y)' = 0$ Ⓐ as this

$f(x) = P'(x) + Q(x)$
$= 0 + x = x$

Ⓐ become

$$y'' + (xy)' = 0$$

$$y' + xy = c_1$$

$$ye^{x^2/2} = \int c_1 e^{x^2/2} + c_2$$

$$y = \frac{c_1}{e^{x^2/2}} \int e^{x^2/2} dx + c_2 e^{-x^2/2}$$

## M-II Reduction of order

Suppose one soln $y_1(x)$, not zero every where is known. Assume second soln

$$y_2(x) = v(x)\, y_1(x) \quad ②$$

$$y'' + P(x)\, y' + Q(x)\, y = 0 \quad ①$$

$$v'' y_1 + [2y_1' + P(x)\, y_1] v' = 0 \quad ③$$

③ is actually a first order eqn in $v'$.

if $\underline{u = v'}$

$$u' y_1 + [2y_1' + P(x)\, y_1]\, u = 0$$

**ex** $2x^2 y'' + 3xy' - y = 0,\ x > 0,\ y = \frac{1}{x}$ is a soln

$$y'' + \frac{3}{2x} y' - \frac{y}{2x^2} = 0$$

$$\frac{v''}{x} + \left[2\left(\frac{-1}{x^2}\right) + \frac{3}{2x}\left(\frac{1}{x}\right)\right] v' = 0$$

$$v'' + \left[\frac{-2}{x} + \frac{3}{2x}\right] v' = 0$$

$$2xv'' + [-4 + 3]\, v' = 0$$

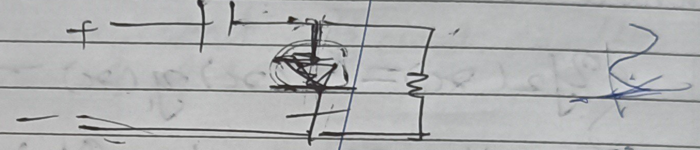
$$2xv'' = v'$$

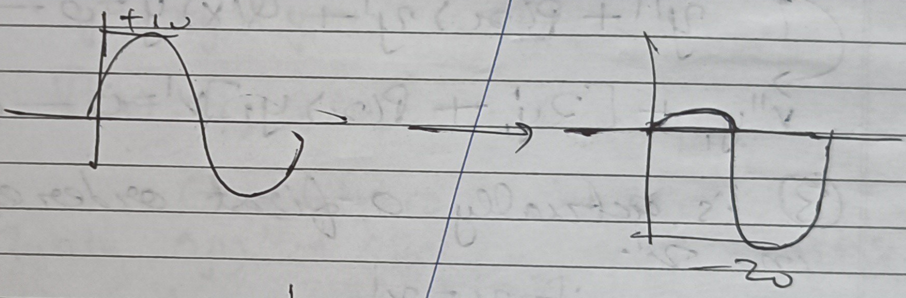
$$2xu' = u$$

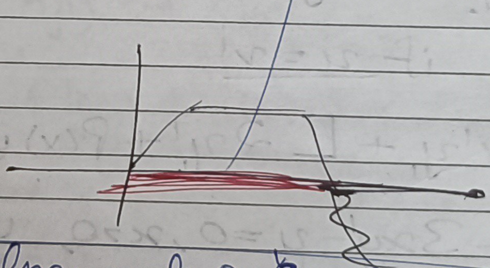
$$\int \frac{u'}{u} = \int \frac{1}{2x}$$

$$I = I_0\left(e^{\frac{Q\boxed{V_d}}{nKT}} - 1\right)$$

Clamper → Capacitor, Resi







$$\ln u = \ln x^{1/2} + \ln K$$

$$u = 2Kx^{1/2}$$

$$\int v' = \int 2Kx^{1/2}$$

$$\boxed{v(x) = \frac{Kx^{3/2}}{3/2} + c}$$

$$\boxed{V(x) = c_1 x^{3/2} + c_2}$$

$$y_2(x) = \frac{2}{3}x^{3/2}$$

## M-II a) Reduction to constant coffeicient

$$ay'' + by' + c = 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}, \quad y = e^{mx}$$

$$ax^2 y'' + bxy' + cy = 0$$

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

$$= c_1 e^{m_1 \ln x} + c_2 e^{m_2 \ln x}$$

$$= c_1 e^{m_1 t} + c_2 e^{m_2 t}, \quad t = \ln x$$

ex

$$x^2 y'' - 2xy' + y = \ln x$$

$$t = \ln x$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{x}\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= \frac{1}{x} \Big| \frac{-1}{x^2} \Big.$$

$$= \frac{1}{x}\frac{d^2y}{dt^2}\frac{1}{x}$$

$$= \frac{1}{x^2}\frac{d^2y}{dt^2}$$

demon → he can find fault with anything

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$$\frac{d^2y}{dt^2} + \frac{F''(x)+p(x)F'(x)}{\{F'(x)\}^2}\frac{dy}{dt} + \frac{q(x)}{\{F'(x)\}^2}y = 0$$

③ has to be with constant coffeicient then

$$\frac{q(x)}{\{F'(x)\}^2} = \text{Constant} = 1$$

$$\Rightarrow f'(x) = q^{1/2}$$

$$f(x) = \int q^{1/2}\,dx$$

ex $y'' - y'\cos x + \sin^2 x\, y = 0$

$$p(x) = -\cos x, \quad q(x) = \sin^2 x$$

$$\frac{q'(x) + 2p(x)q(x)}{q^{3/2}(x)} = \frac{2\sin x\cos x - 2\sin x\cos x = 0}{\sin^3 x}$$

$$t = \int q^{1/2}\,dx = -\cos x$$

$t = F(x) = \int q^{1/2}(x)\,dx;\ q > 0$

Changes ① into eqn with constant coffeicient iff $\frac{q'(x) + 2p(x)q(x)}{q^{3/2}(x)}$ is constant